

A nonlinear singular perturbation problem

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Abstract

Let

$$F(u_\varepsilon) + \varepsilon(u_\varepsilon - w) = 0 \quad (1)$$

where F is a nonlinear operator in a Hilbert space H , $w \in H$ is an element, and $\varepsilon > 0$ is a parameter. Assume that $F(y) = 0$, and $F'(y)$ is not a boundedly invertible operator. Sufficient conditions are given for the existence of the solution to (1.1) and for the convergence $\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - y\| = 0$. An example of applications is considered. In this example F is a nonlinear integral operator.

1 Introduction

In many physical problems the behavior of a solution to an equation depending on a small parameter is of interest. There is a large literature on this topic ([CH],[L], [VT]). The novel point in this paper is the treatment of such a problem for a nonlinear operator equation in a Hilbert space without the usual assumption that the Fréchet derivative of the nonlinear operator at the solution of the limiting equation is a Fredholm-type linear operator.

In a real Hilbert space consider a nonlinear operator $F \in C_{loc}^3$, i.e., $\sup_{u \in B(u_0, R)} \|F^{(j)}(u)\| \leq M_j$, $j = 1, 2, 3$, where $M_j = M_j(R)$ are constants, $u_0 \in H$ is some element, $R > 0$ is a number, $F^{(j)}(u)$ are Fréchet derivatives, and $B(u_0, R) = \{u : \|u - u_0\| \leq R\}$.

Assume that $F(y) = 0$ and $F(u_\varepsilon) + \varepsilon(u_\varepsilon - w) = 0 \ \forall \varepsilon > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$, $w \in H$ is an element. We are interested in the conditions under which $\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - y\| = 0$. This question has been studied much in the literature ([CH],[VT]) under the following assumptions:

- i) $\exists F'(y) := A$

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ii) A is an isomorphism of H onto H ,
or, in place of ii) one may make a weaker assumption:
iii) $A \in \text{Fred}(H)$, i.e., A is a Fredholm-type operator,
that is, the range $R(A)$ is closed, the null-space $N(A)$ is finite-dimensional, $\dim N(A) = n < \infty$, and $\dim N(A^*) = n^* < \infty$.

One may define $z_\varepsilon := u_\varepsilon - y$, $z_0 = 0$, $F(y + z_\varepsilon) := \phi(z_\varepsilon)$, and consider the following equation:

$$\phi(z_\varepsilon) + \varepsilon z_\varepsilon + \varepsilon(y - w) = 0. \quad (1.1)$$

The problem is to prove (under suitable assumptions) that

$$\lim_{\varepsilon \rightarrow 0} \|z_\varepsilon\| = 0. \quad (1.2)$$

One has $A := F'(y) = \phi'(0)$. If the above assumptions i) and ii) hold, then it is known (see [VT]) that equation (1.1) has a unique solution z_ε for all $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 > 0$ is sufficiently small, and (1.2) holds. Indeed, using the Taylor formula, one gets:

$$\phi(z_\varepsilon) = Az_\varepsilon + K(z_\varepsilon), \quad \|K(z_\varepsilon)\| \leq \frac{M_2 \|z_\varepsilon\|^2}{2}. \quad (1.3)$$

One writes (1.1) as

$$z_\varepsilon = -A_\varepsilon^{-1}K(z_\varepsilon) - \varepsilon A_\varepsilon^{-1}(y - w), \quad A_\varepsilon := A + \varepsilon I, \quad (1.4)$$

and applies the contraction mapping theorem to (1.4). This yields the existence of a solution z_ε to (1.1) for all $\varepsilon \in (0, \varepsilon_0)$ and the convergence result (1.2) in a stronger form $\|z_\varepsilon\| = O(\varepsilon)$. The application of the contraction mapping theorem is possible because of the estimate $\sup_{\varepsilon \in (0, \varepsilon_0)} \|A_\varepsilon^{-1}\| \leq c$, which is easy to prove if $\|A^{-1}\| \leq c$, that is, if assumptions i) and ii) hold.

If $A \in \text{Fred}(H)$ then there is also a bifurcation theory for equation (1.1), but it is more complicated than in the case when A is an isomorphism of H onto H (see [VT] and [CH]).

The main novel point of our paper is a study of equation (1.1) in the case when A is not a Fredholm-type operator.

For example, A can be a compact operator.

Our basic result is a proof of relation (1.2) under the assumption

$$\|(A + \varepsilon)^{-1}\| \leq c\varepsilon^{-1}, \quad \varepsilon > 0, c = \text{const}, \quad (1.5)$$

where c does not depend on ε , and w is suitably chosen.

Condition (1.5) holds if there exists a set $\{\zeta \in \mathbb{C} : |\zeta| < r, \pi - \alpha < \arg \zeta < \pi + \alpha\}$ which consists of regular points of the operator $A := F'(y)$. Here $r > 0$ and $\alpha > 0$ are arbitrary small positive numbers. In particular, if $A \geq 0$ is a selfadjoint operator, then (1.5) holds with $c = 1$.

As an application of Theorem 2.1, Section 2, we study the following integral equation:

$$\int_D g(x, s) u_\varepsilon^3(s) ds + \varepsilon(u_\varepsilon - w) = f, \quad g(x, s) := (4\pi|x - s|)^{-1}, \quad f \in C_0^\infty(D). \quad (1.6)$$

Under suitable assumptions on f we prove (1.2) with y being a solution to equation

$$\int_D \frac{y^3(s) ds}{4\pi|x - s|} = f. \quad (1.7)$$

In Section 2 Theorem 2.1 is formulated and proved.

In Section 3 equation (1.6) is studied.

2 Formulation and proof of the result

Theorem 2.1. *Assume that $F \in C_{loc}^3$, $F(y) = 0$, F is compact, (1.5) holds with $A := F'(y)$, and w is such that $y - w = Av$, where $\|v\| < \frac{1}{2M_2c(1+c)}$, and $c > 0$ is the constant in (1.5). Then (1.1) has a solution for all $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is sufficiently small, (1.2) holds, and $\|z_\varepsilon\| = O(\varepsilon)\|$. The solution to (1.1) is unique in a sufficiently small ball $\{u : \|u - y\| \leq R\}$, $R = O(\varepsilon)$.*

Remark 2.1. *If $\overline{R(A)} = H$, then one can always find a w such that $y - w = Av$ and $\|v\| < b$, where $b > 0$ is an arbitrary small number. This conclusion holds under much weaker assumption. Namely, let A_b denote the restriction of A to the ball $B(0, b) := \{u : \|u\| \leq b\}$, where $b = \text{const} > 0$ is the radius of the ball. Let $R_b := \{v : v = Au, u \in B(0, b)\}$, and let $\overline{R_b}$ be its closure. If $\overline{R_b} \cap \{B(0, r) \setminus \{0\}\} \neq \emptyset$, where $r > 0$ is some number, then there exists a w such that $y - w = Av$, $\|v\| \leq b$.*

Proof. Rewrite equation (1.1) as (1.4). Denote the right-hand side of (1.4) by $T(z_\varepsilon)$. Let $B(R) := \{z : \|z\| \leq R\}$. Choose a suitable dependence $R = R(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ (see (2.2)). Then $TB(R) \subset B(R)$ if ε is sufficiently small.

Indeed,

$$A_\varepsilon^{-1}(y - w) = A_\varepsilon^{-1}Av = v - \varepsilon A_\varepsilon^{-1}v,$$

so

$$\|A_\varepsilon^{-1}(y - w)\| \leq \|v\| + c\|v\|,$$

and

$$\|T(z)\| \leq \| - A_\varepsilon^{-1}K(z) - \varepsilon A_\varepsilon^{-1}(y - w) \| \leq \frac{c M_2 R^2}{\varepsilon} + \varepsilon \|v\| (1 + c) \leq R, \quad (2.1)$$

provided that

$$\frac{\varepsilon}{cM_2}(1 - \rho) \leq R \leq \frac{\varepsilon}{cM_2}(1 + \rho), \quad (2.2)$$

where $\rho = \sqrt{1 - 2M_2\|v\|c(1+c)}$.

Since F is compact, so are K and $T := -A_\varepsilon^{-1}K - \varepsilon A_\varepsilon^{-1}(y - w)$. As we have proved above, the operator T maps the ball $B(0, R)$ into itself if ε is sufficiently small. Therefore, by the Schauder's fixed-point theorem, the map T has a fixed point in $B(0, R)$, i.e., equation (1.4) has a solution in $B(0, R)$. Since $R = O(\varepsilon)$, it follows that $\|z_\varepsilon\| = O(\varepsilon)$, so (1.2) holds.

To prove uniqueness of the solution to (1.1), it is sufficient to prove uniqueness of the solution to (1.4) in the ball $B(0, R) := \{z_\varepsilon : \|z_\varepsilon\| \leq R\}$. If (1.4) has two solutions, say z and v , then their difference $p := z - v$ solves the equation $p = -A_\varepsilon^{-1}[K(z) - K(v)]$, where

$$K(z) := \int_0^1 (1-s)F''(y + sz)z \, ds$$

is the remainder term in the Taylor formula $F(u_\varepsilon) - F(y) = Az + K(z)$. If $\|z\| \leq R$ and $\|v\| \leq R$, then one has

$$\|K(z) - K(v)\| \leq \int_0^1 ds(1-s)(M_3R^2s + 2M_2R)\|p\| = \left(\frac{M_3R^2}{6} + M_2R\right)\|p\|. \quad (2.3)$$

Thus, using (??), one gets:

$$\|p\| \leq q\|p\|, \quad q := \frac{c}{\varepsilon}\left(\frac{M_3R^2}{6} + M_2R\right). \quad (2.4)$$

Take $R = \frac{\varepsilon(1-\rho)}{cM_2}$ (see (2.2)). Then $q \leq 1 - \rho + \varepsilon \frac{M_3(1-\rho)^2}{6cM_2^2} < 1$ if ε is sufficiently small. Thus, $p = 0$ if ε is sufficiently small. Theorem 2.1 is proved. \square

3 An example

Consider equation (1.6). Let $D \subset \mathbb{R}^n$ be a bounded domain, $n = 3$, and

$$F(u) := \int_D g(x, s)u^3(s)ds := Gu^3, \quad f = 0, \quad F'(u)\psi = 3 \int_D g(x, s)u^2(s)\psi(s)ds.$$

Then $y(s) = 0$, $F(y) = 0$, $F'(y) = 0$, and

$$\|(F'(y) + \varepsilon)^{-1}\| = \frac{1}{\varepsilon},$$

so that (1.5) holds. We took $f = 0$ for simplicity.

Let $\|u\|_{H^1(D)} := \|u\|_1$, $\|u\|_{L^2(D)} := \|u\|_0$. Let us check that equation (1.6) with $w = \varepsilon h$, where $\|h\|_1 = 1$, h is otherwise arbitrary, has a unique solution u_ε for any $\varepsilon \in (0, \varepsilon_0)$. Write (1.6) as

$$u_\varepsilon = -\frac{1}{\varepsilon}F(u_\varepsilon) + \varepsilon h := T(u_\varepsilon). \quad (3.1)$$

Let us check that $TB_1(R) \subset B_1(R) := \{u : \|u\|_1 \leq R\}$, and T is a contraction mapping on $B(R)$, where $R = \varepsilon^{2/3}$. As a Hilbert space we take $H = H^1(D)$, the Sobolev space.

If $n = 3$ then $\|u\|_{L^6(D)} \leq c\|u\|_{H^1(D)}$ by the embedding theorem, so $u^3 \in L^2(D)$ and

$$\|Gu^3\|_{H^2(D)} = \left\| \int_D g(x, s)u^3(s)ds \right\|_{H^2(D)} \leq c\|u^3\|_0$$

by the properties of the Newtonian potential. Thus

$$\|F(u)\|_{H^2(D)} \leq c\|u\|_{L^6(D)}^3 \leq c_1\|u\|_{H^1(D)}^3,$$

where by c and c_1 various positive constants, independent of u , are denoted. One has $\|h\|_1 = 1$, and $\|T(u)\|_1 \leq \frac{c_1}{\varepsilon}\|u\|_1^3 + \varepsilon$. If $\|u\|_1 \leq R$, then $\|T(u)\|_1 \leq R$ provided that

$$\frac{c_1}{\varepsilon}R^3 + \varepsilon \leq R. \quad (3.2)$$

Choose $R = \varepsilon^{2/3}$. Then (3.2) holds if $c_1\varepsilon + \varepsilon \leq \varepsilon^{2/3}$. This inequality holds for all $\varepsilon \in (0, \varepsilon_0)$ if $\varepsilon_0 > 0$ is sufficiently small, namely $\varepsilon_0 < (1 + c_1)^{-3}$.

Thus $T : B_1(R) \rightarrow B_1(R)$ if $R = \varepsilon^{2/3}$ and $\varepsilon \in (0, \varepsilon_0)$.

Let us check that T is a contraction on $B_1(R)$. If $u, z \in B_1(R)$, then one has

$$\|T(u) - T(z)\|_1 = \frac{1}{\varepsilon}\|F(u) - F(z)\|_1 \leq \frac{1}{\varepsilon}\|G(u^3 - z^3)\|_1.$$

Furthermore,

$$\|G(u^3 - z^3)\|_1 \leq c\|u^3 - z^3\|_0 \leq c\|u - z\|_0(\|u\|_{L^4(D)}^2 + \|z\|_{L^4(D)}^2) \leq c_2\|u - z\|_1 R^2.$$

Thus, with $R = \varepsilon^{2/3}$, one has:

$$\|T(u) - T(z)\|_1 \leq \frac{c_2 R^2}{\varepsilon}\|u - z\|_1 \leq c_2 \varepsilon^{1/3}\|u - z\|_1.$$

If $\varepsilon \in (0, \varepsilon_0)$ and $\varepsilon_0^{1/3}c_2 < 1$, then T is a contraction map on $B(R)$ in the space $H^1(D)$.

Therefore we have proved the following:

Theorem 3.1. *Assume $\|h\|_1 = 1$ and $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is sufficiently small. Then equation (3.1) has a unique solution u_ε and $\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_1 = 0$.*

References

- [CH] Chow, S., Hale, J., **Methods of bifurcation theory**, Springer Verlag, New York, 1982.
- [L] Lomov, S., **Introduction to the general theory of singular perturbations**, Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1992
- [VT] Vainberg, M., Trenogin, V., **The theory of branching solutions of nonlinear equations**, Nauka, Moscow, 1969.